Ordering and measuring actuarial risks

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1. Introduction: sums of r.v.’s

- Many problems in risk theory involve sums of r.v.’s:
  \[ S = X_1 + X_2 + \cdots + X_n \]

- Standard techniques for (approx.) evaluation of the d.f. of \( S \):
  Convolution, moment-based approximations, recursions.

- Assuming independence of the \( X_i \) is often not appropriate:
  - Introducing stochastic financial aspects.
  - Non-independence of remaining lifetimes.

- The copula approach: (Frees & Valdez, 1998)
  \[ \Pr [X_1 \leq x_1, \ldots, X_n \leq x_n] = C [F_{X_1}(x_1), \ldots, F_{X_n}(x_n)] \]
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- **Problem to solve:**
  - Summarize $S = X_1 + X_2 + \ldots + X_n$ into a real number $\rho[S]$.
  - Suppose: $F_{X_i}$ known, $n$ large, $C$ complicated / unknown.

- **How to solve?**
  - Derive stochastic lower and upper bounds for $S$:
    \[
    S^l \preceq S \preceq S^u
    \]
  - Approximate $\rho[S]$ by $\rho[S^u]$ or $\rho[S^l]$.
  - **Required tools:**
    - Risk measures
    - Convex order
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- **How to solve?**
  - Derive stochastic lower and upper bounds for $S$:
    \[ S^l \lesssim S \lesssim S^u \]
  - Approximate $\rho[S]$ by $\rho[S^u]$ or $\rho[S^l]$.

- **Required tools:**
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2. Risk measures

2.1. General

Definition:
A risk measure is a mapping from a set of losses to the real line:

\[ X \rightarrow \rho [X] \]

Have been investigated extensively in the literature:

- Huber (1981):
  Upper expectations.
- Goovaerts, De Vylder & Haezendonck (1984):
  Premium principles.
- Artzner, Delbaen, Eber & Heath (1999):
  Coherent risk measures.
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2. Risk measures

2.2. Distortion risk measures

- $\bar{F}_X(x) = \Pr[X > x]$.
- $E[X] = I - \Pi$. 

![Diagram showing the relationship between $\bar{F}_X(x)$ and the expected value $E[X]$]
2. Risk measures

2.2. Distortion risk measures

- \( \overline{F}_X(x) = \Pr [X > x] \).
- \( E[X] = I - II \).
2. Risk measures

2.2. Distortion risk measures (cont’d)

- **Distortion function:**
  
  \( g(x) \) is nondecreasing, \( g(0) = 0 \) and \( g(1) = 1 \).

- **Distortion risk measure:** \( \rho_g[X] \overset{\text{def.}}{=} (I + I') - II \).

- In case \( g(x) \geq x \): \( \rho_g[X] \geq E[X] \).
2. Risk measures

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- **In case** \( g(x) \geq x \): \[ \rho_g [X] \geq E[X]. \]
2. Risk measures

2.3. Value-at-Risk

- VaR\(_p[X]\) = \(F_X^{-1}(p) = Q_p[X]\).
- Distortion function:

\[ g(x) = 1 \ (x > 1 - p) \]
2. Risk measures

2.3. Value-at-Risk

- \( \text{VaR}_p[X] = F_X^{-1}(p) = Q_p[X] \).
- **Distortion function:**

\[
g(x) = \mathbb{1}(x > 1 - p)
\]
2. Risk measures

2.4. Tail Value-at-Risk

- TVaR$_p (X) = \frac{1}{1-p} \int_p^1 \text{VaR}_q [X] \ dq$.
- Distortion function:

\[ g(x) = \min \left( \frac{x}{1-p}, 1 \right) \]
2. Risk measures

2.4. Tail Value-at-Risk

- **TVaR}_p(X) = \frac{1}{1-p} \int_p^1 \text{VaR}_q[X] \ dq.
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2. Risk measures

2.5. Concave distortion risk measures

- \( \rho_g [X] = \int_0^1 \text{VaR}_{1-q} [X] \ dg(q) \).
- \( \rho_g \) is a concave distortion risk measure if \( g \) is concave.
- TVaR\( p \) is concave, VaR\( p \) not.
- Concave distortion risk measures are subadditive:

\[
\rho_g [X + Y] \leq \rho_g [X] + \rho_g [Y]
\]

- Optimality of VaR\( p \): (Artzner et al. 1999)
  \[
  \text{VaR}_p [X] = \inf \{ \rho([X] | \rho \text{ is coherent and } \rho \geq \text{VaR}_p) \}
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- Optimality of TVaR\( p \):
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  \text{TVaR}_p [X] = \min \{ \rho_g ([X] | g \text{ is concave and } \rho_g \geq \text{VaR}_p) \}
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2.5. Concave distortion risk measures

- $\rho_g[X] = \int_0^1 \text{VaR}_1 - q[X] \, dg(q)$.
- $\rho_g$ is a concave distortion risk measure if $g$ is concave.
- TVaR$_p$ is concave, VaR$_p$ not.
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- Optimality of VaR$_p$: (Artzner et al. 1999)

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3. Optimality of VaR
(D., Goovaerts & Kaas, 2003)

- Consider a loss \( X \) and a solvency capital requirement \( \rho [X] \).

- The insolvency risk and the cost of insolvency:
  \[
  (X - \rho [X])_+ \quad \rightarrow \quad E \left[ (X - \rho [X])_+ \right]
  \]

- The cost of capital:
  \( \rho [X] \times i \)

- How to choose \( \rho [X] \)?
  - \( E \left[ (X - \rho [X])_+ \right] \) should be small \( \Rightarrow \) \( \rho [X] \) large.
  - Capital has a cost \( \Rightarrow \) \( \rho [X] \) small.
3. Optimality of VaR
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- Consider a loss $X$ and a solvency capital requirement $\rho[X]$.
- The insolvency risk and the cost of insolvency:

$$ (X - \rho[X])_+ \rightarrow \mathbb{E} [(X - \rho[X])_+] $$

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- Consider a loss $X$ and a solvency capital requirement $\rho [X]$.
- The insolvency risk and the cost of insolvency:

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- How to choose $\rho [X]$?
  - $\mathbb{E} [(X - \rho [X])_+]$ should be small $\Rightarrow \rho [X]$ large.
  - Capital has a cost $\Rightarrow \rho [X]$ small.
3. Optimality of VaR (cont’d)

- The optimal capital requirement: \( \rho[X] \) is determined as the minimizer (with respect to \( d \)) of

\[
E[(X - d)_+] + d \varepsilon, \quad 0 < \varepsilon < 1
\]

- Solution:

\[
\rho[X] = \text{VaR}_{1-\varepsilon}[X]
\]

- The minimum of the cost function is given by \( \varepsilon \text{ TVaR}_{1-\varepsilon}[X] \).

- Geometric proof (for \( \text{VaR}_{1-\varepsilon}[X] > 0 \)): 

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- Geometric proof (for $\text{VaR}_{1-\varepsilon}[X] > 0$):
3. Optimality of VaR (cont’d)

- $\mathbb{E}[(X - d)_+] + d \varepsilon$ in case $d = Q_{1-\varepsilon}[X]$:
3. Optimality of VaR (cont’d)

- \( E[(X - d)_+ ] + d \varepsilon \) in case \( d < Q_{1-\varepsilon} [X] \):
3. Optimality of VaR (cont’d)

- \( E[(X - d)_+] + d \epsilon \) in case \( d > Q_{1-\epsilon}[X] \):
4. Can a risk measure be too subadditive?
(D., Laeven, Vanduffel, Darkiewicz & Goovaerts, 2008)

- In case the capital requirement $\rho$ is additive:

$$ (X + Y - \rho[X + Y])_+ \leq (X - \rho[X])_+ + (Y - \rho[Y])_+ $$

- Splitting increases the insolvency risk.

  - $\rho$ should be subadditive:

  $$ \rho[X + Y] \leq \rho[X] + \rho[Y] $$

- Merging decreases the insolvency risk.

  - Subadditivity of $\rho$ is allowed to some extent.
  - Without any restriction, $\rho$ could be too subadditive.

- In case the capital requirement $\rho$ is additive:

$$
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$$

- **Splitting** increases the insolvency risk.
  - $\rho$ should be subadditive:

$$
\rho[X + Y] \leq \rho[X] + \rho[Y]
$$

- **Merging** decreases the insolvency risk.
  - Subadditivity of $\rho$ is allowed *to some extent.*
  - Without any restriction, $\rho$ could be *too subadditive.*
4. Can a risk measure be too subadditive?

(D., Laeven, Vanduffel, Darkiewicz & Goovaerts, 2008)

- In case the capital requirement \( \rho \) is additive:

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4. Can a risk measure be too subadditive? (cont’d)

- The regulator’s condition for the capital requirement $\rho$:

$$
E \left[ (X + Y - \rho[X + Y])_+ \right] + \varepsilon \rho[X + Y] \\
\leq E \left[ (X - \rho[X])_+ \right] + E \left[ (X - \rho[X])_+ \right] + \varepsilon (\rho[X] + \rho[Y])
$$

- $\text{VaR}_{1-\varepsilon}[\cdot]$ fulfills the regulator’s condition.
- Any subadditive $\rho[\cdot] \geq \text{VaR}_{1-\varepsilon}[\cdot]$ fulfills the regulator’s condition.
- Markowitz, 1959:
  'We might decide that in one context one basic set of principles is appropriate, while in another context a different set of principles should be used.'
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5. Comonotonicity

5.1. General

- **Definition:**
  \((X_1, \cdots, X_n)\) is **comonotonic** if there exists a r.v. \(Z\) and increasing functions \(f_1, \cdots, f_n\) such that
  \[
  (X_1, \cdots, X_n) \overset{d}{=}(f_1(Z), \ldots, f_n(Z))
  \]

- (\(X_1, \cdots, X_n\)) has a 1-dimensional stochasticity.
- Comonotonicity is very strong positive dependency structure.
- Adding comonotonic r.v.'s:
  - Produces no diversification.
  - If all \(X_i\) are identically distributed and comonotonic, then
    \[
    \frac{X_1 + \cdots + X_n}{n} \overset{d}{=} X_1
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5. Comonotonicity

5.2. An example

- Consider \((X, Y, Z)\) with
  \(X \sim \text{Uniform on } (0, \frac{1}{2}) \cup (1, \frac{3}{2})\)
  \(Y \sim \text{Beta (2,2)}\)
  \(Z \sim \text{Normal (0,1)}\).

- Support of \((X, Y, Z)\) when they are mutually independent:
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5.2. An example (cont’d)

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5.3. Sums of comonotonic r.v.'s

- **Notation:**
  - \((X_1^c, \ldots, X_n^c) = \text{comonotonic counterpart of } (X_1, \ldots, X_n)\).
  - \(S^c = X_1^c + X_2^c + \cdots + X_n^c\).

- **Quantiles of** \(S^c\):
  \[
  F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p)
  \]

- **Distribution function of** \(S^c\):
  \[
  \sum_{i=1}^{n} F_{X_i}^{-1} [F_{S^c}(x)] = x
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5.3. Sums of comonotonic r.v.’s (cont’d)

- **Stop-loss premiums of** $S^c$: (D., Wang, Young, Goovaerts, 2000)

\[
E [S^c - d]_+ = \sum_{i=1}^{n} E [(X_i - d_i)_+] 
\]

with

\[
d_i = F_{X_i}^{-1} [F_{S^c} (d)]
\]

- **Jamshidian’s formula**: (Jamshidian, 1989)
  - Assume the Vasicek (1977) model.
  - Price of a European call on a coupon bond
    \[= \text{sum of prices of European calls on zero coupon bonds}\].
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6. An allocation problem

(D., Goovaerts, Kaas, 2003; D., Tsanakas, Valdez, Vanduffel, 2009)

Problem description:

- Consider the loss portfolio \((X_1, \ldots, X_n)\).
- \(X_i = \) loss of business unit \(i\).
- \(d = \) aggregate solvency capital.
- How to measure the performance of the business units?
  - Allocate \(d\) among the \(n\) business units.
  - Determine the returns on the allocated capitals.

 Allocation rule:

\[
\min \sum_{i=1}^{n} d_i = d \quad \text{subject to} \quad \sum_{i=1}^{n} (X_i - d_i)_+ \leq d
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6. An allocation problem
(D., Goovaerts, Kaas, 2003; D., Tsanakas, Valdez, Vanduffel, 2009)

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► **Allocation rule:**

\[
\min_{\sum_{i=1}^{n} d_i = d} \mathbb{E} \left( \sum_{i=1}^{n} \left[ (X_i - d_i)_+ \right] \right)
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\min_{\sum_{i=1}^{n} d_i = d} \mathbb{E} \left( \sum_{i=1}^{n} [(X_i - d_i)_+] \right)
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6. An allocation problem (cont’d)

Solution of the minimization problem:

- Let \( S = X_1 + \cdots + X_n \) and \( S^c = X_1^c + \cdots + X_n^c \).
- For all \( d_i \) with \( \sum_{i=1}^n d_i = d \), we have

\[
E \left[ (S^c - d)_+ \right] \leq \sum_{i=1}^n E \left[ (X_i - d_i)_+ \right]
\]

- Notice that

\[
E \left[ (S^c - d)_+ \right] = \sum_{i=1}^n E \left[ \left( X_i - F_{X_i}^{-1} \left[ F_{S^c}(d) \right] \right)_+ \right]
\]

- Also notice that

\[
\sum_{i=1}^n F_{X_i}^{-1} [F_{S^c}(d)] = d
\]

- Conclusion: the optimal allocation is given by

\[
d_i^* = F_{X_i}^{-1} [F_{S^c}(d)]
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6. An allocation problem (cont’d)

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$$E[(S^c - d)_+] \leq \sum_{i=1}^{n} E[(X_i - d_i)_+]$$

- Notice that

$$E[(S^c - d)_+] = \sum_{i=1}^{n} E\left[(X_i - F_{X_i}^{-1}[F_{S^c}(d)])_+\right]$$

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$$d_i^* = F_{X_i}^{-1}[F_{S^c}(d)]$$
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6. An allocation problem (cont’d)

- Solution of the minimization problem:
  - Let \( S = X_1 + \cdots + X_n \) and \( S^c = X_1^c + \cdots + X_n^c \).
  - For all \( d_i \) with \( \sum_{i=1}^{n} d_i = d \), we have
    \[
    \mathbb{E}[(S^c - d)_+] \leq \sum_{i=1}^{n} \mathbb{E}[(X_i - d_i)_+]
    \]

- Notice that
  \[
  \mathbb{E}[(S^c - d)_+] = \sum_{i=1}^{n} \mathbb{E}\left[\left(X_i - F_{X_i}^{-1}[F_{S^c}(d)]\right)_+\right]
  \]

- Also notice that
  \[
  \sum_{i=1}^{n} F_{X_i}^{-1}[F_{S^c}(d)] = d
  \]

- Conclusion: the optimal allocation is given by
  \[
  d_i^* = F_{X_i}^{-1}[F_{S^c}(d)]
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6. An allocation problem (cont’d)

Solution of the minimization problem:

- Let $S = X_1 + \cdots + X_n$ and $S^c = X_1^c + \cdots + X_n^c$.
- For all $d_i$ with $\sum_{i=1}^{n} d_i = d$, we have

\[
E[(S^c - d)_+] \leq \sum_{i=1}^{n} E[(X_i - d_i)_+]
\]

Notice that

\[
E[(S^c - d)_+] = \sum_{i=1}^{n} E \left[ \left( X_i - F_{X_i}^{-1}[F_{S^c}(d)] \right)_+ \right]
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\[
d^*_i = F_{X_i}^{-1}[F_{S^c}(d)]
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7. Convex order

7.1 Upper and lower tails of d.f.'s

\[ E[(X - d)_+] = \text{surface above the d.f., from } d \text{ on:} \]
7. Convex order

7.1 Upper and lower tails of d.f.'s (cont'd)

- \( E[ (d - X)_+ ] \) = surface below the d.f., from \(-\infty\) to \(d\).
7. convex order

7.2. Defining convex order

- **Definition:**

\[ X \leq_{cx} Y \iff \text{any tail of } Y \text{ exceeds the corresponding tail of } X \]

- Risk averse decision makers prefer loss \( X \) over loss \( Y \).

- **Characterization in terms of distortion risk measures:**
  (Wang & Young, 1998; Denuit et al., 2005)

\[ X \leq_{cx} Y \iff \mathbb{E}[X] = \mathbb{E}[Y] \text{ and } \rho_g [X] \leq \rho_g [Y] \text{ for all concave } g. \]
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7. Convex order

7.3. Convex bounds for sums of r.v.'s

orem: (Kaas et al., 2000)
For any \((X_1, \cdots, X_n)\) and any \(\Lambda\), we have that

\[
\sum_{i=1}^{n} E[X_i \mid \Lambda] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} X_i^c
\]

Notation: \(S^l \leq_{cx} S \leq_{cx} S^c\).

When all \(E[X_i \mid \Lambda]\) are \(\uparrow\) functions of \(\Lambda\):

- \(S^l\) is a comonotonic sum.

Why use these comonotonic bounds?

- One-dimensional stochasticity.
- \(\rho_g[S^l]\) and \(\rho_g[S^c]\) are easy to calculate.
- If \(g\) is concave, then \(\rho_g[S^l] \leq \rho_g[S] \leq \rho_g[S^c]\).
7. Convex order

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\]

- **Notation**: \(S^I \leq_{cx} S \leq_{cx} S^c\).

- When all \(\mathbb{E}[X_i \mid \Lambda]\) are \(\uparrow\) functions of \(\Lambda\):
  - \(S^I\) is a comonotonic sum.

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  - One-dimensional stochasticity.
  - \(\rho_g[S^I]\) and \(\rho_g[S^c]\) are easy to calculate.
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- **When all \(\mathbb{E}[X_i \mid \Lambda]\) are \(\nearrow\) functions of \(\Lambda\):**
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7. Convex order

7.4 On the choice of the conditioning r.v.

(Kaas et al., 2000; Vanduffel et al., 2005, 2009)

- Consider the multivariate normal random vector \((Y_1, \cdots, Y_n)\).
- The random sum \(S\) and its approximation \(S^l\):

\[
S = \sum_{i=1}^{n} e^{Y_i} \quad \text{and} \quad S^l = \sum_{i=1}^{n} E \left[ e^{Y_i} \mid \Lambda \right]
\]

- First order approximation for \(\text{Var}[S^l]\):

\[
\text{Var}[S^l] \approx \left( \text{Corr} \left[ \sum_{i=1}^{n} E[e^{-Y_i}] Y_i, \Lambda \right] \right)^2 \text{Var} \left[ \sum_{i=1}^{n} E[e^{-Y_i}] Y_i \right]
\]

- Optimal choice for \(\Lambda\):

\[
\Lambda = \sum_{i=1}^{n} E \left[ e^{Y_i} \right] Y_i
\]
7. Convex order

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(Kaas et al., 2000; Vanduffel et al., 2005, 2009)

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\[
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\]

- First order approximation for \(\text{Var}[S']\):

\[
\text{Var}[S'] \approx \left( \text{Corr} \left( \sum_{i=1}^{n} \mathbb{E}[e^{-Y(i)}] Y_i, \Lambda \right) \right)^2 \text{Var} \left[ \sum_{i=1}^{n} \mathbb{E}[e^{-Y(i)}] Y_i \right]
\]

- Optimal choice for \(\Lambda\):

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\Lambda = \sum_{i=1}^{n} \mathbb{E}\left[ e^{Y_i} \right] Y_i
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8. Exotic options
(Simon, Goovaerts & D. 2000; Chen, Deelstra, D. & Vanmaele, 2008)

- **A European style basket call option:**
  - Consider a basket of \( n \) stocks (dividend-paying or not).
  - \( A_i(t) = \) price of stock \( i \) at time \( t \geq 0 \).
  - \( T = \) exercise date of the option.
  - \( K = \) exercise price.
  - Pay-off of the call option:
    \[
    \text{Pay-off at } T = \left( \sum_{i=1}^{n} w_i A_i(T) - K \right) +
    \]

- **Assumptions:**
  - Risk free interest rate = \( \delta \).
  - No arbitrage.
  - Arbitrage-free time 0 price of the basket option:
    \[
    BC[K, T] = e^{-\delta T} \mathbb{E} \left[ \left( \sum_{i=1}^{n} w_i A_i(T) - K \right) + \right]
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  - \( T \) = exercise date of the option.
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  - Pay-off of the call option:

    \[
    \text{Pay-off at } T = \left( \sum_{i=1}^{n} w_i A_i(T) - K \right)_+ \]

- **Assumptions:**
  - Risk free interest rate = \( \delta \).
  - No arbitrage.
  - Arbitrage-free time 0 price of the basket option:

    \[
    BC[K, T] = e^{-\delta T} \mathbb{E} \left[ \left( \sum_{i=1}^{n} w_i A_i(T) - K \right)_+ \right] 
    \]

- Expectations and probabilities are evaluated wrt \( Q \).
8. Exotic options

8.2. A comonotonic upperbound

Define $S$ and $S^c$ by:

$$S = \sum_{i=1}^{n} w_i A_i(T) \text{ and } S^c = \sum_{i=1}^{n} w_i A^c_i(T)$$

An upper bound for the basket option price:

$$BC[K, T] = e^{-\delta T} E[(S - K)_+] \leq e^{-\delta T} E[(S^c - K)_+]$$

European call options on stock $i$:

$$EC_i[k, T] = e^{-\delta T} E[(A_i(T) - k)_+]$$

The upper bound in terms of European call option prices:

$$e^{-\delta T} E[(S^c - K)_+] = \sum_{i=1}^{n} w_i EC_i\left[F_{A_i(T)}^{-1}\left(F_{S^c(nK)}\right), T\right]$$
8. Exotic options
8.2. A comonotonic upperbound

Define $S$ and $S^c$ by:

$$S = \sum_{i=1}^{n} w_i A_i(T) \text{ and } S^c = \sum_{i=1}^{n} w_i A^c_i(T)$$

**An upper bound for the basket option price:**

$$BC[K, T] = e^{-\delta T} \mathbb{E} [(S - K)_+] \leq e^{-\delta T} \mathbb{E} [(S^c - K)_+]$$

**European call options on stock $i$:**

$$EC_i[k, T] = e^{-\delta T} \mathbb{E} [(A_i(T) - k)_+]$$

**The upper bound in terms of European call option prices:**

$$e^{-\delta T} \mathbb{E} [(S^c - K)_+] = \sum_{i=1}^{n} w_i EC_i \left[ F_{A_i(T)}^{-1} (F_{S^c}(nK)), T \right]$$
8. Exotic options

8.2. A comonotonic upperbound

- Define $S$ and $S^c$ by:

$$S = \sum_{i=1}^{n} w_i \ A_i(T) \quad \text{and} \quad S^c = \sum_{i=1}^{n} w_i \ A^c_i(T)$$

- An upper bound for the basket option price:

$$BC[K, T] = e^{-\delta T} \ E \left[(S - K)_+\right] \leq e^{-\delta T} \ E \left[(S^c - K)_+\right]$$

- European call options on stock $i$:

$$EC_i[k, T] = e^{-\delta T} \ E \left[(A_i(T) - k)_+\right]$$

- The upper bound in terms of European call option prices:

$$e^{-\delta T} \ E \left[(S^c - K)_+\right] = \sum_{i=1}^{n} w_i \ EC_i \left[F^{-1}_{A_i(T)}(F_{S^c}(nK)), T\right]$$
8. Exotic options

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BC[K, T] = e^{-\delta T} \ E[(S - K)_+] \leq e^{-\delta T} \ E[(S^c - K)_+]
$$

- European call options on stock $i$:

$$
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- The upper bound in terms of European call option prices:

$$
e^{-\delta T} \ E[(S^c - K)_+] = \sum_{i=1}^{n} w_i \ EC_i \left[ F_{A_i(T)}^{-1} \left( F_{S^c(nK)} \right), T \right]
$$
8. Exotic options
8.3. Buying combinations of European calls

- **The investment strategy** \( (\nu_1, \nu_2, \ldots, \nu_n) \):
  - At time 0, we buy a combination of European calls \( \text{EC}_i [k, T] \) for different \( i \) and \( k \).
  - Pay-off at time \( T \):
    \[
    \sum_{i=1}^{n} \int_{0}^{\infty} (A_i(T) - k)_+ \ d\nu_i(k)
    \]
  - Any such investment strategy is static and characterized by
    \[
    \nu = (\nu_1, \nu_2, \ldots, \nu_n)
    \]

- **Price at time 0 of investment strategy** \( \nu \):
  \[
  \text{Price} [\nu] = \sum_{i=1}^{n} \int_{0}^{\infty} \text{EC}_i [k, T] \ d\nu_i(k)
  \]
8. Exotic options

8.3. Buying combinations of European calls

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    \[\nu = (\nu_1, \nu_2, \ldots, \nu_n)\]

- Price at time 0 of investment strategy \(\nu\):
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  - **Pay-off at time** \(T\):
    \[
    \sum_{i=1}^{n} \int_{0}^{\infty} (A_i(T) - k)_+\, d\nu_i(k)
    \]
  - Any such investment strategy is static and characterized by
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  \text{Price}[\nu] = \sum_{i=1}^{n} \int_{0}^{\infty} EC_i[k, T] \, d\nu_i(k)
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8.3. Buying combinations of European calls

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    \]
  - Any such investment strategy is static and characterized by
    \[
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    \]

- Price at time 0 of investment strategy \(\nu\):
  \[
  \text{Price} [\nu] = \sum_{i=1}^{n} \int_{0}^{\infty} EC_i[k, T] \, d\nu_i(k)
  \]
8. Exotic options

8.3. Buying combinations of European calls (cont’d)

▶ **Super-replicating strategies:**

▶ Consider the strategies \( \nu \) such that for all \( x \geq 0 \) it holds that:

\[
\sum_{i=1}^{n} \int_{0}^{+\infty} (x_i - k)_+ \, d\nu_i(k) \geq \left( \sum_{i=1}^{n} w_i \, x_i - K \right)_+
\]

▶ Denote the set of such strategies by \( \mathcal{A}_K \).

▶ The price of \( \nu \in \mathcal{A}_K \) exceeds the price of the basket option:

\[
\text{Price}[\nu] \geq \text{BC}(K, T)
\]
8. Exotic options
8.3. Buying combinations of European calls (cont’d)

Super-replicating strategies:

Consider the strategies $\nu$ such that for all $x \geq 0$ it holds that:

$$
\sum_{i=1}^{n} \int_{0}^{+\infty} (x_i - k)_+ \, d\nu_i(k) \geq \left( \sum_{i=1}^{n} w_i \, x_i - K \right)_+
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The price of $\nu \in \mathcal{A}_K$ exceeds the price of the basket option:

$$\text{Price } [\nu] \geq \text{BC}(K, T)$$
8. Exotic options

8.3. Buying combinations of European calls (cont’d)

- **Super-replicating strategies:**

  - Consider the strategies $\nu$ such that for all $x \geq 0$ it holds that:
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  - Denote the set of such strategies by $\mathcal{A}_K$.

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8. Exotic options
8.3. Buying combinations of European calls (cont’d)

- Super-replicating strategies:
  - Consider the strategies \( \nu \) such that for all \( x \geq 0 \) it holds that:
    \[
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    \]
  - Denote the set of such strategies by \( A_K \).

- The price of \( \nu \in A_K \) exceeds the price of the basket option:
  \[
  \text{Price}[\nu] \geq BC(K, T)
  \]
8. Exotic options

8.4. The cheapest super-replicating strategy

- **Upperbounds for the basket option:**
  - The comonotonic upperbound:
    \[
    BC(K, T) \leq e^{-\delta T} E \left[(S^c - K)_+\right]
    \]
  - The price of any \( \nu \in \mathcal{A}_K \) is an upperbound for \( BC(K, T) \).

- **Theorem**
  - The comonotonic UB is the price of the cheapest super-replicating strategy in \( \mathcal{A}_K \):
    \[
    e^{-\delta T} E \left[(S^c - K)_+\right] = \min_{\nu \in \mathcal{A}_K} \sum_{i=1}^{n} \int_{0}^{+\infty} EC_i [k, T] \, d\nu_i(k)
    \]
  - The \( \nu \) corresponding with the comonotonic UB:
    - For each \( i \), buy \( w_i \) European calls \( EC_i \{ F_{K^i} T \} (F_{Sc}(nK), T) \).
8. Exotic options

8.4. The cheapest super-replicating strategy

- **Upperbounds for the basket option:**
  
  - The comonotonic upperbound:
    
    \[
    BC(K, T) \leq e^{-\delta T} E\left[(S^c - K)_+\right]
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Upperbounds for the basket option:

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\[ BC(K, T) \leq e^{-\delta T} E \left[ (S^c - K)_+ \right] \]

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Theorem

The comonotonic UB is the price of the cheapest super-replicating strategy in \( \mathcal{A}_K \):

\[ e^{-\delta T} E \left[ (S^c - K)_+ \right] = \min_{\nu \in \mathcal{A}_K} \sum_{i=1}^{n} \int_{0}^{+\infty} \nu_i(k) \, EC_i[k, T] \, d\nu_i(k) \]

The \( \nu \) corresponding with the comonotonic UB:

For each \( i \), buy \( \nu_i \) European calls \( EC_i \left[ F_{\Delta+1} \left( F_{c}(nK) \right), T \right] \).
8. Exotic options
8.4. The cheapest super-replicating strategy

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    \]
  - The \( \nu \) corresponding with the comonotonic UB:
    - For each \( i \), buy \( w_i \) European calls \( EC_i \left[ F_{A_i(T)}(F_{S^c}(nK)), T \right] \).
8. Exotic options
8.4. The cheapest super-replicating strategy

Upperbounds for the basket option:

The comonotonic upperbound:

$$BC(K, T) \leq e^{-\delta T} E \left[ (S^c - K)_+ \right]$$

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Theorem

The comonotonic UB is the price of the cheapest super-replicating strategy in $A_K$:

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The $\nu$ corresponding with the comonotonic UB:

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8.4. The cheapest super-replicating strategy

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    - For each \( i \), buy \( w_i \) European calls \( EC_i\left[F_{A_i(T)}^{-1}(F_{S^c}(nK)), T\right]. \)
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- **Upperbounds for the basket option:**
  - The comonotonic upperbound:
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    \]
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    - For each \( i \), buy \( w_i \) European calls \( EC_i \left[ F_{A_i(T)}(F_{S^c}(nK)), T \right] \).
8. Exotic options
8.5. A worst-case expectation

- The Fréchet class $\mathcal{R}_n$:

$$\mathcal{R}_n = \left\{ F_Y \mid F_{Y_i}(x) = F_{A_i(T)}(x); x \geq 0, i = 1, \ldots, n \right\}$$

- Theorem:

The comonotonic UB is the worst-case expectation in $\mathcal{R}_n$:

$$e^{-\delta T} E \left[ (S^c - K)_+ \right] = \max_{F_Y \in \mathcal{R}_n} e^{-\delta T} E \left[ \left( \sum_{i=1}^{n} w_i Y_i - K \right)_+ \right]$$
8. Exotic options

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8. Exotic options
8.6. Remarks and generalizations

- The upper bound is "model-free":
  - It can be determined for any given model of the stock prices.
  - It can be determined from observed European call prices.

- Asian call options:
  - Pay-off at time $T$:
    \[
    \text{Pay-off} = \left( \frac{1}{n} \sum_{i=0}^{n-1} A(T - i) - K \right)_+ 
    \]
  - Price at time 0:
    \[
    AC[K, T] = e^{-\delta T} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=0}^{n-1} A(T - i) - K \right)_+ \right] 
    \]
  - Upper bound:
    \[
    AC[K, T] \leq e^{-\delta T} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=0}^{n-1} A^c(T - i) - K \right)_+ \right] 
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8.6. Remarks and generalizations

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  - Pay-off at time $T$:
    
    $$\text{Pay-off} = \left( \frac{1}{n} \sum_{i=0}^{n-1} A(T - i) - K \right)_+$$

  - **Price at time 0:**
    
    $$AC[K, T] = e^{-\delta T} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=0}^{n-1} A(T - i) - K \right)_+ \right]$$

  - **Upper bound:**
    
    $$AC[K, T] \leq e^{-\delta T} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=0}^{n-1} A^c(T - i) - K \right)_+ \right]$$
8. Exotic options
8.6. Remarks and generalizations

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  - Price at time 0:
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  - Upper bound:
    \[
    AC[K, T] \leq e^{-\delta T} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=0}^{n-1} A^c(T - i) - K \right)_+ \right] \]
8. Exotic options
8.6. Remarks and generalizations (cont’d)

► **Generalization to the finite market case.**
   (Hobson et al., 2005; Chen et al., 2008).

► **Available European calls on stock $i$:**

![Diagram of available European calls on stock $i$.]
8. Exotic options

8.6. Remarks and generalizations (cont’d)

- **Generalization to the finite market case.**
  (Hobson et al., 2005; Chen et al., 2008).

- **Available European calls on stock $i$:**

![Diagram showing available European calls on stock $i$]
Exotic options

8.7. Numerical illustration: Asian call options

- Risk-free interest rate \( e^{\delta} - 1 = 9\% \) per year.
- \( \{A(t)\} \): geometric Brownian motion with \( A(0) = 100 \) and volatility per year \( \sigma = 0.2 \).
- \( n = 10 \) days, \( T = \text{day 120} \).
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- Let $B(\tau)$ be a standard Wiener process.
- A continuous perpetuity: (Dufresne, 1989; Milevsky, 1997) = an eternal continuous payment of 1 per year.
- The single premium $\pi$ for the continuous perpetuity:
  - $\pi$ is invested in a fund with cumulative return over $(0, \tau)$ given by
    \[ \exp [\mu \tau + \sigma B(\tau)] \]
  - Stochastic present value of the perpetuity liabilities:
    \[ S = \int_0^\infty \exp [-\mu \tau - \sigma B(\tau)] \, d\tau \]
  - $\pi = Q_p[S]$ with $p$ sufficiently large.
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The continuous perpetuity (cont’d)

- **Numerical illustration:** \( \mu = 0.07 \) and \( \sigma = 0.1 \).
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![Graph showing numerical illustration](image)
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