Optimal Dividends: Analysis with Brownian Motion

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Abstract

In the absence of dividends, the surplus of a company is modeled by a Wiener process (or Brownian motion) with positive drift. Now dividends are paid according to a barrier strategy: whenever the (modified) surplus attains the level b, the "overflow" is paid as dividends to shareholders. An explicit expression for the moment generating function of the time of ruin is given. Let D denote the sum of the discounted dividends until ruin. Explicit expressions for the expectation and the moment generating function of D are given; furthermore, the limiting distribution of D is determined, when the variance parameter of the surplus process tends to infinity. It is shown that the sum of the (undiscounted) dividends until ruin is a compound geometric random variable with exponentially distributed summands. The optimal level b* is the value of b for which the expectation of D is maximal. It is shown that b* is an increasing function of the variance parameter; as the variance parameter tends to infinity, b* tends to the ratio of the drift parameter and the valuation force of interest, which can be interpreted as the present value of a perpetuity. The leverage ratio is the expectation of D divided by the initial surplus invested; it is observed that this leverage ratio is a decreasing function of the initial surplus. For b = b*, the expectation of D, considered as a function of the initial surplus, has the properties of a risk averse utility function, as long as the initial surplus is less than b*. The expected utility of D is calculated for quadratic and exponential utility functions. In an appendix, the original discrete model of De Finetti is explained and a probabilistic identity is derived.
1. Introduction

Traditionally, actuaries have been primarily concerned with the financial management of insurance companies and other financial systems, in particular with their solvency. In the classical model for determining the probability ruin, the surplus of a company can increase without bounds. This is unrealistic. De Finetti (1957) suggested that other, more economic considerations such as dividend payments should also play an important role. Specifically, he considered a discrete-time model, in which the periodic gains of a company are +1 (with probability $\pi > 1/2$) or −1 (with probability $1 - \pi$). If the ultimate goal is to maximize the expectation of the discounted dividends paid to the shareholders of the company, what is the optimal dividend-payment strategy? De Finetti found that the optimal strategy must be a barrier strategy, and he showed how the optimal level of the barrier can be determined.

Subsequently, the problem of finding the optimal dividend-payment strategy was discussed extensively by Karl Borch; see Borch (1974, 1990). The reader may also want to consult the monographs Bühlmann (1970, Section 6.4), Gerber (1979, Sections 10.1 and 10.2) and Seal (1969, pp. 163-166), and their references. Some recent papers are Paulsen and Gjessing (1997), Siegl and Tichy (1999), Claramunt, Mármol and Alegre (2002), and Gerber and Shiu (2002a, b). The reader should be cautioned that in more general models the optimal strategy can be a band strategy and not a barrier strategy.

In this paper we go back to the roots: we consider the continuous counterpart of De Finetti’s model. Here it is assumed that the surplus of a company is a Wiener process (Brownian motion) with a positive drift. This model has the advantage that some very explicit calculations can be made. Furthermore, the economic analysis of the results can
be carried out more easily in considerable depth. Basic mathematical results can be found in Gerber (1972), where the surplus process is the sum of a Wiener process and an independent compound Poisson process; but the paper provides no economic analyses.

2. The Wiener Process Model and Basic Results

Consider a company with initial surplus or equity \( x > 0 \). If no dividends were paid, the surplus at time \( t \) would be

\[
X(t) = x + \mu t + \sigma W(t), \quad t \geq 0, \quad (2.1)
\]

with \( \mu > 0, \sigma > 0, \) and \( \{W(t)\} \) being a standard Wiener process. This model can be found in Section 6.9 of Klugman, Panjer and Willmot (1998), which is a textbook for the current Examinations 3 and 4. The company will pay dividends to its shareholders according to a barrier strategy with parameter \( b > 0 \). Whenever the (modified) surplus reaches the level \( b \), the “overflow” will be paid as dividends. A formal definition can be given in terms of the running maximum

\[
M(t) = \max_{0 \leq \tau \leq t} X(\tau). \quad (2.2)
\]

Then the aggregate dividends paid by time \( t \) are

\[
D(t) = (M(t) - b)_+ = \begin{cases} 0 & \text{if } M(t) \leq b, \\ M(t) - b & \text{if } M(t) > b. \end{cases} \quad (2.3)
\]

See Figure 1. The modified surplus at time \( t \) is \( X(t) - D(t) \).
Let $\delta > 0$ be the force of interest for valuation, and let $D$ denote the present value of all dividends until ruin,

$$D = \int_0^T e^{-\delta t} dD(t),$$

(2.4)

where

$$T = \min\{t \geq 0 \mid X(t) - D(t) = 0\}$$

(2.5)

is the time of ruin. We use the symbol $V(x; b), 0 \leq x \leq b$, for the expectation of $D$,

$$V(x; b) = E[D].$$

(2.6)

As a function of initial surplus $x$, $V(x; b)$ satisfies the homogeneous second-order differential equation

$$\frac{\sigma^2}{2} V''(x; b) + \mu V'(x; b) - \delta V(x; b) = 0, \quad 0 \leq x \leq b.$$  

(2.7)

This is best seen by the following heuristic argument. Let $0 < x < b$. In the infinitesimal time interval from 0 to $dt$, the surplus, with $X(0) = x$, does not reach either barrier (0 or $b$). Hence
\[
E[V(X(dt); b)] = e^{\delta dt} V(x; b). \tag{2.8}
\]

The right-hand side of (2.8) is
\[
(1 + \delta dt)V(x; b) = V(x; b) + \delta V(x; b) dt.
\]

Since
\[
X(dt) = x + \mu dt + \sigma W(dt),
\]
the left-hand side of (2.8) is
\[
V(x; b) + \mu V'(x; b) dt + \frac{\sigma^2}{2} V''(x; b) dt.
\]

Thus subtracting \( V(x; b) \) from both sides of (2.8) and then canceling \( dt \) yields (2.7).

The function \( V(x; b) \) satisfies the boundary conditions
\[
V(0; b) = 0, \tag{2.9}
\]
\[
V'(b; b) = 1. \tag{2.10}
\]

Condition (2.9) is obvious: if \( X(0) = x = 0 \), ruin is immediate, and no dividends are paid.

Condition (2.10) can be explained as follows: consider two situations, one with initial surplus \( x = b \), and the other with initial surplus \( x = b - \epsilon \) (\( \epsilon > 0 \) and “small”). Then in the first situation, the dividends will be by the amount \( \epsilon \) higher than in the second case, for almost all sample paths of \( \{W(t)\} \).

Subject to the boundary conditions (2.9) and (2.10), the solution of the differential equation (2.7) is
\[
V(x; b) = \frac{g(x)}{g'(b)}, \quad 0 \leq x \leq b, \tag{2.11}
\]
where
\[
g(x) = e^{rx} - e^{sx}, \tag{2.12}
\]
with \( r \) and \( s \) being the roots of the quadratic equation
\[
\frac{\sigma^2}{2} \xi^2 + \mu \xi - \delta = 0. \tag{2.13}
\]

We let \( r \) denote the positive root and \( s \) the negative root,

\[
r = -\mu + \frac{\sqrt{\mu^2 + 2\delta \sigma^2}}{\sigma^2}, \tag{2.14}
\]

\[
s = -\mu - \frac{\sqrt{\mu^2 + 2\delta \sigma^2}}{\sigma^2}, \tag{2.15}
\]

so that both the numerator and the denominator in (2.11) are positive.

We can rewrite (2.11) as

\[
V(x; b) = \frac{g(x)}{g(b)} \frac{g(b)}{g(b)} V(b; b), \quad 0 \leq x \leq b. \tag{2.16}
\]

Then we see that the ratio,

\[
\frac{g(x)}{g(b)} = \frac{e^{rx} - e^{sx}}{e^{rb} - e^{sb}}, \quad 0 \leq x \leq b, \tag{2.17}
\]

can be interpreted as the expected discounted value of a contingent payment of 1, payable as soon as the surplus reaches level \( b \), provided ruin has not yet occurred. Formula (2.17) is equivalent to (10.13.15) in Panjer et al. (1998).

Remarks (i) Let \( \tau > 0 \) denote a fixed time, \( u = b - x \), and

\[
B(u, \tau) = E\left[ \int_0^\tau e^{-\delta t} dD(t) \right] \tag{2.18}
\]

be the expectation of the present value of the dividends between 0 and \( \tau \). Gerber and Pafumi (1998, Theorem 1) have found a closed-form formula for \( B(u, \tau) \) and Gerber and
Shiu (2003, (2.10)) have extended it to the case of negative $\delta$. The formula has three terms:

$$B(u, \tau) = -\frac{\mu}{\delta} e^{-\delta u} \Phi\left(\frac{-u + \mu \tau}{\sigma \sqrt{\tau}}\right)$$

$$- \frac{\sigma^2}{2\delta} e^{-\delta u} \Phi\left(\frac{-u - (r - s)\sigma \sqrt{\tau}}{2}\right)$$

$$- \frac{\sigma^2}{2\delta} e^{-\delta u} \Phi\left(\frac{-u + (r - s)\sigma \sqrt{\tau}}{2}\right).$$

(2.19)

Note that, by the law of iterated expectations,

$$V(x; b) = E\{B(b - x; T)\}. \quad (2.20)$$

The expectation is taken with respect to the time-of-ruin random variable $T$ for $X(0) = 0$. However, (2.20) is not a practical way to arrive at (2.11)!

(ii) In the remainder of this section, we consider the limiting case $\delta = 0$. Then $D = D(T)$, the total dividends paid until ruin, and $V(x; b) = E\{D(T)\}$. From (2.14) and (2.15), we get $r = 0$ and $s = -2\mu/\sigma^2$. Hence

$$g(x) = 1 - e^{-2\mu x/\sigma^2} \quad (2.21)$$

and

$$V(x; b) = \frac{\sigma^2}{2\mu} \left(e^{2\mu b/\sigma^2} - e^{2\mu (b-x)/\sigma^2}\right), \quad 0 \leq x \leq b, \quad (2.22)$$

by (2.11). In particular,

$$V(b; b) = \frac{\sigma^2}{2\mu} \left(e^{2\mu b/\sigma^2} - 1\right), \quad 0 \leq b \leq b. \quad (2.23)$$

It follows from (2.22) and (2.23) that

$$V(b; b) = V(b - x; b - x) + V(x; b), \quad 0 \leq x \leq b. \quad (2.24)$$
This formula can be interpreted as follows. For $X(0) = b$, the total dividends paid until ruin can be decomposed as the sum of the total dividends paid until the modified surplus drops to the level $x$ for the first time and the total dividends paid thereafter until ruin. Taking expectations we obtain (2.24). In the limit $\mu \to 0$ (in addition to $\delta = 0$), (2.22) becomes

$$V(x; b) = E[D(T)] = x,$$

(2.25)

which is independent of $b$ and $\sigma$.

From (2.17) and (2.21), we see that

$$\frac{g(x)}{g(b)} = \frac{1 - e^{-2\mu x/\sigma^2}}{1 - e^{-2\mu b/\sigma^2}}, \quad 0 \leq x \leq b. \quad (2.26)$$

This is the probability that the Wiener process $\{X(t)\}$, with $X(0) = x$, $0 \leq x \leq b$, will reach level $b$ before level $0$. Formula (2.26) is well known; for example, it can be found in Karlin and Taylor (1975, Chapter 7, Theorem 5.2) and Harrison (1985, page 43) and it is equivalent to (10.13.20) in Panjer et al. (1998). The discrete counterpart of (2.26) can be found in some textbooks in the context of gambler's ruin problem. In the limiting case $\delta = 0$ and $\mu = 0$, (2.26) is

$$\frac{g(x)}{g(b)} = \frac{x}{b}, \quad 0 \leq x \leq b. \quad (2.27)$$

3. The Distribution of $T$ under a Barrier Strategy

Consider that the barrier strategy with level $b$ is applied. Thus ruin is certain. We are interested in the distribution of the time of ruin, $T$. In this section, we calculate

$$L(x; b) = E[e^{-\delta T}], \quad (3.1)$$
where $x = X(0)$ is the initial surplus or capital. This is the expected present value of a payment of 1 at the time of ruin, and at the same time, the Laplace transform of the probability density function of $T$. We shall also determine the expected time to ruin, $E[T]$.

The functions $L(x; b)$, $V(x; b)$ and (2.17) are special cases of a family of functions $\{K(x; b)\}$, where

$$K(x; b) = E[e^{-\delta \tau} K(X(\tau); b)], \quad 0 \leq x \leq b, \quad (3.2)$$

with

$$\tau = \min \{t \geq 0 \mid X(t) = 0 \text{ or } X(t) = b\} \quad (3.3)$$

being the first time when the surplus attains the level $b$ or falls to 0. The argument we used to derive (2.7) also shows that $K(x; b)$ satisfies the homogeneous second-order differential equation

$$\frac{\sigma^2}{2} K''(x; b) + \mu K'(x; b) - \delta K(x; b) = 0, \quad 0 \leq x \leq b. \quad (3.4)$$

Hence the function $K(x; b)$ is a linear combination of the exponential functions $e^{rx}$ and $e^{sx}$, with $r$ and $s$ given by (2.14) and (2.15), respectively. The coefficients of this linear combination depend on the boundary conditions. For $L(x; b)$, the boundary conditions are

$$L(0; b) = 1, \quad (3.5)$$

$$L'(b; b) = 0. \quad (3.6)$$

It follows that

$$L(x; b) = \frac{re^{-s(b-x)} - se^{-r(b-x)}}{re^{-sb} - se^{-rb}}, \quad 0 \leq x \leq b. \quad (3.7)$$
This expression can be found in Example 5.6 of Cox and Miller (1965, p. 233), which also indicates how the probability density function of $T$ can be obtained by inverting (3.7).

Let

$$m(y) = E[e^{yT}]$$

(3.8)
denote the moment generating function of $T$. Then $m(-\delta)$ is $L(x; b)$ of above. Thus, from (3.7), (2.14) and (2.15), we have

$$m(y) = \frac{r(y)e^{-s(y)(b-x)} - s(y)e^{-r(y)(b-x)}}{r(y)e^{-s(y)b} - s(y)e^{-r(y)b}}, \quad 0 \leq x \leq b,$$

(3.9)

with

$$r(y) = -\frac{\mu + \sqrt{\mu^2 - 2y\sigma^2}}{\sigma^2},$$

(3.10)

$$s(y) = -\frac{\mu - \sqrt{\mu^2 - 2y\sigma^2}}{\sigma^2}.$$  

(3.11)

The moments of $T$ can be derived by differentiation. For example, noting that

$$r(0) = 0, \quad s(0) = -\frac{2\mu}{\sigma^2}, \quad r'(0) = -\frac{1}{\mu}, \quad s'(0) = \frac{1}{\mu},$$

we get

$$E[T] = m'(0)$$

$$= \frac{\sigma^2}{2\mu^2} \left[ e^{\frac{\mu b}{\sigma^2}} - e^{\frac{\mu (b-x)}{\sigma^2}} - \frac{2\mu x}{\sigma^2} \right], \quad 0 \leq x \leq b.$$  

(3.12)

It is instructive to write (3.12) as

$$E[T] = \frac{1}{\mu} [E[D(T)] - x], \quad 0 \leq x \leq b,$$  

(3.13)
by applying (2.22). Formula (3.13) can be derived directly by means of the optional sampling theorem. Consider the martingale \{X(t) – µt\}. Stopping it at the time of ruin, \( T \), and applying the optional sampling theorem yields

\[
x = E[X(T) – µT].
\] (3.14)

Now,

\[
X(T) = D(T)
\] (3.15)

by (2.5) (the modified surplus being 0 at the time of ruin). Thus (3.14) is the same as (3.13).

The cumulant generating function

\[
c(y) = \ln[m(y)]
\] (3.16)

provides additional insight for the roles of the initial surplus \( x \) and the barrier level \( b \).

From (3.9), we see that

\[
c(y) = h(y; b – x) – h(y; b),
\] (3.17)

where

\[
h(y; \xi) = \ln[r(y)e^{-s(y)\xi} – s(y)e^{-r(y)\xi}].
\] (3.18)

Then the k-th cumulant of \( T \) is

\[
c^{(k)}(0) = h^{(k)}(0; b – x) – h^{(k)}(0; b).
\] (3.19)

In particular,

\[
E[T] = c'(0)
\]

\[
= h'(0; b – x) – h'(0; b),
\]

where

\[
h'(0; \xi) = -\frac{\sigma^2}{2\mu^2}[e^{2\mu\xi/\sigma^2} – 1 - \frac{2\mu\xi}{\sigma^2}].
\] (3.20)

This confirms (3.12). It is interesting to rewrite (3.20) as
\[ h'(0; \xi) = -\frac{1}{\mu} [V(\xi; \xi)_{\xi=0} - \xi], \quad \xi \geq 0, \quad (3.21) \]

by applying (2.23).

**Remarks** (i) There is an unexpected relation between \( V'(0; b) \) and \( L(b; b) \). From (2.11) and (2.12), we see that

\[ V'(0; b) = \frac{r-s}{re^{rb} - se^{sb}}. \quad (3.22) \]

From (3.7) we gather that

\[ L(b; b) = \frac{r-s}{re^{-rb} - se^{-sb}} \]

\[ = e^{(r+s)b} \frac{r-s}{re^{rb} - se^{sb}} \]

\[ = e^{(r+s)b} V'(0; b). \quad (3.23) \]

Noting that

\[ r + s = -\frac{2\mu}{\sigma^2}, \quad (3.24) \]

we obtain from (3.23) the surprising identity

\[ L(b; b) = e^{-2\mu b/\sigma^2} V'(0; b). \quad (3.25) \]

Now,

\[ L(b; b) = \int_0^\infty e^{-\delta} \Pr[t < T \leq t + dt \mid X(0) = b], \quad (3.26) \]

\[ V'(0; b) = \frac{d}{dx} \int_0^\infty e^{-\delta} E[I(T > t) \mid D(t + dt) - D(t) \mid X(0) = x] \mid_{x=0}. \quad (3.27) \]

Here \( I(A) \) denotes the indicator random variable of an event \( A \). Because (3.25) is valid for all \( \delta > 0 \), it follows that
\[
\frac{e^{2\mu b/\sigma^2}}{\sqrt{2\pi} \sigma} \Pr[t < T \leq t + dt \mid X(0) = b] = \left. \frac{d}{dx} E[I(T > t) [D(t + dt) - D(t)] \mid X(0) = x] \right|_{x=0}.
\] (3.28)

In Appendix B, we present the discrete counterparts of identities (3.25) and (3.28). Some readers will find that (B.22) is easier to understand than (3.28).

(ii) Consider the limit \( \sigma \to \infty \). Noting that \( r \) and \( s \) tend to 0, we gather from (2.12) that, for \( \sigma \to \infty \),

\[
g(x) \sim (r - s)x \quad (3.29)
\]

and

\[
g'(x) \sim r - s. \quad (3.30)
\]

Applying (3.29) and (3.30) to (2.11), we see that, for \( 0 \leq x \leq b \),

\[
V(x; b) \to x \quad \text{as} \quad \sigma \to \infty,
\] (3.31)

independently of \( b, \mu \) and \( \delta \). Furthermore, the limit of (3.12) for \( \sigma \to \infty \) is 0. Because \( T \) is a positive random variable, we conclude that its limiting distribution is the degenerate distribution at 0. Loosely speaking, the interpretation of these results is as follows: in the case of infinite risk, ruin is practically instantaneous, and the expectation of the dividends before ruin is equal to the initial surplus. That the latter depends neither on \( \delta \) nor on \( \mu \) is explained by the fact that ruin occurs “instantaneously.” Formula (3.31) exhibits the limit of the expectation of the random variable \( D \). In Section 4 more insight will be provided: we shall determine the limit of the distribution of \( D \).

(iii) Consider the deterministic case \( \sigma = 0 \). Because \( \mu > 0 \), ruin will not occur, \( T \equiv \infty \).

For \( 0 \leq x \leq b \), it takes \((b - x)/\mu\) units of time for the surplus to reach level \( b \). Then dividends will be paid continuously and perpetually at rate \( \mu \) per unit time. Hence
\[ V(x; b) = e^{-\delta(b-x)/\mu} \frac{\mu}{\delta}, \quad 0 \leq x \leq b. \] (3.32)

It is easy to check that (3.32) satisfies the differential equation (2.7) with \( \sigma = 0 \) and condition (2.10); condition (2.9) is not applicable because \( \mu > 0 \).

4. The Moment Generating Function of \( D \)

If the barrier strategy with barrier level \( b \) is applied, the present value of the resulting dividends until ruin, \( D \), is a random variable. Its expectation is given by (2.11). However, one might be interested in more detailed information concerning the distribution of \( D \), for example, the higher order moments of \( D \). This section examines the moment generating function of \( D \),

\[ M(x, y; b) = E[e^{yD}]. \] (4.1)

To obtain a functional equation for \( M(x, y; b) \), assume \( 0 < X(0) = x < b \). Then

\[ M(x, y; b) = E[M(X(dt), e^{-\delta dt}; b)]. \] (4.2)

By expanding the last expression, we obtain after simplification the partial differential equation

\[ \frac{\sigma^2}{2} \frac{\partial^2 M}{\partial x^2} + \mu \frac{\partial M}{\partial x} - \delta y \frac{\partial M}{\partial y} = 0, \] (4.3)

which generalizes (2.7). Furthermore, the boundary conditions are

\[ M(0, y; b) = 1 \] (4.4)

and

\[ \frac{\partial M(x, y; b)}{\partial x} \bigg|_{x=b} = y M(b, y; b), \] (4.5)

which generalize (2.9) and (2.10), respectively.
To solve (4.3)–(4.5), we let

\[ V_k(x; b) = E[D^k], \quad k = 1, 2, 3, \ldots (4.6) \]

Note that \( V_1(x; b) = V(x; b) \). Then

\[
M(x, y; b) = 1 + \sum_{k=1}^{\infty} \frac{y^k}{k!} E[D^k] = 1 + \sum_{k=1}^{\infty} \frac{y^k}{k!} V_k(x; b), (4.7)
\]

substitution of which in (4.3) and comparing the coefficients of \( y^k \) yields the ordinary differential equations

\[
\sigma^2 \frac{d^2}{dx^2} V_k(x; b) + \mu \frac{d}{dx} V_k(x; b) - \delta k V_k(x; b) = 0, \quad k = 1, 2, 3, \ldots (4.8)
\]

It follows from (4.4) that

\[
V_k(0; b) = 0, \quad k = 1, 2, 3, \ldots , (4.9)
\]

and from (4.5) that

\[
V'_1(b; b) = 1, (4.10)
\]

which is (2.10), and that

\[
V'_k(b; b) = kV_{k-1}(b; b), \quad k = 2, 3, 4, \ldots (4.11)
\]

From (4.8) and (4.9), it follows that, for \( k = 1, 2, 3, \ldots \),

\[ V_k(x; b) = C_k(b) g_k(x), (4.12) \]

with

\[
g_k(x) = e^{r_k x} - e^{s_k x}, (4.13)
\]

where \( r_k \) and \( s_k \) are the roots of the equation

\[
\frac{\sigma^2}{2} \beta^2 + \mu \xi - \delta k = 0. (4.14)
\]
To determine the coefficient functions $C_k(.)$, we apply (4.12) to (4.10) and to (4.11). We then obtain

$$C_1(b) = \frac{1}{g'_1(b)}, \quad (4.15)$$

which confirms (2.11), and

$$C_k(b) g'_k(b) = kC_{k-1}(b) g_{k-1}(b), \quad k = 2, 3, 4, \ldots \quad (4.16)$$

Hence

$$C_k(b) = k! \frac{g_1(b) \ldots g_{k-1}(b)}{g'_1(b) \ldots g'_{k-1}(b) g'_k(b)}, \quad (4.17)$$

and the $k$-th moment of $D$ about the origin is

$$V_k(x; b) = k! \frac{g_1(b) \ldots g_{k-1}(b) g_k(x)}{g'_1(b) \ldots g'_{k-1}(b) g'_k(b)}, \quad k = 1, 2, 3, \ldots \quad (4.18)$$

Finally, by (4.7), the moment generating function of $D$ is

$$E[e^{yD}] = M(x, y; b) = 1 + \sum_{k=1}^{\infty} y^k \frac{g_1(b) \ldots g_{k-1}(b) g_k(x)}{g'_1(b) \ldots g'_{k-1}(b) g'_k(b)}. \quad (4.19)$$

**Remark** For $\sigma \to \infty$, the limiting distribution of $D$ can be determined as follows. Note that $r_k \to 0$ and $s_k \to 0$ for $\sigma \to \infty$. Hence, for $\sigma \to \infty$,

$$g_k(x) \sim (r_k - s_k)x$$

and

$$g'_k(x) \sim r_k - s_k.$$ 

It follows from this and (4.19) that, for $y < 1/b$ and $\sigma \to \infty$,

$$E[e^{yD}] \to 1 + \sum_{k=1}^{\infty} y^k b^{k-1} x.$$
Thus the limiting distribution of $D$ is a mixture of the degenerate distribution at 0 and the exponential distribution with mean $b$. The weights of this mixture, $(b - x)/b$ and $x/b$, are respectively the probability of not reaching and the probability of reaching the dividend barrier $b$ before ruin; see (2.27). Formula (3.31) follows from (4.20).

5. The Distribution of $D(T)$

Throughout this section, we consider $\delta = 0$ and $0 < x < b$. Hence in this section, the functions $g(x)$ and $V(x; b)$ are given by (2.21) and (2.22), respectively. Because $D(T) = D$ when $\delta = 0$, the moment generating function of $D(T)$ (which is the same as $X(T)$) can be obtained from (4.19) as the limiting case $\delta \to 0$. For simplicity, we shall not adjust our notation to signify that the interest rate is zero.

From (4.14) we see that $r_k = 0$, $s_k = -2\mu/\sigma^2$, and

$$g_k(x) = g(x) = 1 - e^{-2\mu x/\sigma^2}$$

for all $k$. Thus we obtain from (4.19) and (2.11) that

$$M(x, y; b) = E[e^{yD(T)} | X(0) = x]$$

$$= 1 + \sum_{k=1}^{\infty} y^k \frac{[V(b; b)]^{k-1}}{1 - V(b; b)y} V(x; b)$$

$$= 1 + \frac{V(x; b)y}{1 - V(b; b)y}$$

$$= 1 - \frac{[V(b; b) - V(x; b)]y}{1 - V(b; b)y}.$$
Note that formula (5.2) is of the type (A.4) in Appendix A, with
\[ \alpha = V(b; b) - V(x; b) = V(b - x; b - x) \] (5.3)
by (2.24), and
\[ \beta = V(b; b). \] (5.4)

Hence the distribution of \( D(T) \) is a mixture of the degenerate distribution at 0 and an exponential distribution. By (A.5) the weights of the mixture are
\[ p = \frac{\alpha}{\beta} = 1 - \frac{V(x; b)}{V(b; b)} = 1 - \frac{g(x)}{g(b)} \] (5.5)
and
\[ q = 1 - p = \frac{g(x)}{g(b)}, \] (5.6)
respectively, and the mean of the exponential distribution is
\[ \mu = \frac{\alpha}{p} = \beta = V(b; b). \] (5.7)

In this context, \( p \) is the probability that the surplus does not reach the barrier \( b \) before ruin occurs. With \( \delta = 0 \), \( V(b; b) \) is the expectation \( E[D(T) \mid X(0) = b] \).

Furthermore, we see from (A.1) that \( D(T) \) has a compound geometric distribution,
\[ D(T) = D_1 + D_2 + \ldots + D_N. \] (5.8)
Here, \( N \) represents the number of the visits of the modified surplus at the dividend barrier \( b \) that are separated by visits at the initial level \( x \). Note that
\[ E[N] = \frac{q}{p} = \frac{g(x)}{g(b) - g(x)} \] (5.9)
increases from 0 to $\infty$ as the initial capital $x$ increases 0 to $b$. For $j = 1, 2, 3, \ldots$, the random variable $D_j$ is the contribution of visit $j$ to $D(T)$. The common distribution of the $D_j$’s is exponential with mean

$$\mu = \alpha = V(b - x; b - x)$$

(5.10)

by (A.1) and (5.3). The decomposition of $D(T)$ is illustrated in Figure 2, where $N = 2$.

**Figure 2**: The decomposition of $D(T)$ as a compound geometric random variable
**Remark** Consider $\mu \to 0$ (in addition to $\delta = 0$). Applying (2.25) to the second last line of (5.2) yields

$$M(x, y; b) = 1 + \frac{xy}{1 - by}. \quad (5.11)$$

Note that the right-hand side of (5.11) does not involve $\sigma$ and, more intriguingly, it is identical to the middle line of (4.20), which was obtained by letting $\sigma \to \infty$. This is not a coincidence, and we can explain it by means of operational time. Let

$$\tilde{t} = \sigma^2 t. \quad (5.12)$$

In the terms of the new time scale, the parameters of the model are:

$$\tilde{\sigma} = 1,$$

$$\tilde{\mu} = \frac{\mu}{\sigma^2},$$

$$\tilde{\delta} = \frac{\delta}{\sigma^2}.$$

Thus $\sigma \to \infty$ means that $\tilde{\mu} \to 0$ and $\tilde{\delta} \to 0$.

6. The Optimal Dividend Barrier

For a given initial surplus $X(0) = x$, let $b^*$ denote the optimal value of $b$, that is, the value that maximizes $V(x; b)$, the expectation of $D$. From (2.11), we see that this is the value minimizing $g'(b)$. Hence $b^*$ is the solution of the equation

$$g''(b^*) = 0. \quad (6.1)$$

This leads to
with \( r \) and \( s \) given by (2.14) and (2.15), respectively. Note that the optimal barrier level \( b^* \) does not depend on the initial surplus \( x \).

The optimal value of \( b \) has a geometric characterization. Let \( W(x; b), x > 0 \), be the expectation of \( D \) if the barrier strategy with parameter \( b \) is applied. Thus
\[
W(x; b) = \begin{cases} 
V(x; b) & \text{if } 0 \leq x \leq b \\
-x + V(b; b) & \text{if } x > b 
\end{cases}.
\] (6.3)

At the junction \( x = b \), the function \( W(x; b) \) is continuous and has a continuous first derivative by (2.10). Under what condition is also the second derivative continuous, that is, \( V''(b; b) = 0 \)? From (2.11), we see that
\[
V''(x; b) = \frac{g''(x)}{g'(b)}.
\] (6.4)

Hence \( V''(b; b) = 0 \) is equivalent to the condition that \( g''(b) = 0 \), which in turn means that \( b = b^* \) by (6.1). This geometric characterization of the optimal parameter value is known as a high contact condition in the finance literature and smooth pasting condition in the optimal stopping literature.

7. Analysis of the Optimal Barrier

If the initial surplus is at the optimal barrier, \( x = b = b^* \), the differential equation (2.7) becomes
\[
\frac{\sigma^2}{2} 0 + \mu 1 - \delta V(b^*; b^*) = 0
\]
because of the condition \( V''(b^*; b^*) = 0 \) and condition (2.10). Hence

\[
V(b^*; b^*) = \frac{\mu}{\delta}. \tag{7.1}
\]

This formula has been obtained by Gerber (1972). At first sight, (7.1) is a surprising result, since \( \mu/\delta \) does not depend on \( \sigma \) and is identical to the present value of a perpetuity-certain with continuous payments at a rate of \( \mu \). In the deterministic case \( \sigma = 0 \), ruin does not occur and, for each \( b > 0 \),

\[
V(b, b) = \frac{\mu}{\delta};
\]

see (3.32) with \( x = b \). However, for the general case \( \sigma > 0 \), ruin does occur and the dividends stop at the time of ruin.

For a better understanding of (7.1), observe that \( b^* \) is the initial surplus necessary to obtain an expected total return of \( \mu/\delta \). Since the latter is independent of \( \sigma \), the necessary initial surplus \( b^* \) must be a function of \( \sigma \) to compensate for the risk. In fact, we now show that

(I) \( b^* \) is an increasing function of \( \sigma \); \hspace{1cm} (7.2)

(II) \( b^* \uparrow \frac{\mu}{\delta} \) for \( \sigma \uparrow \infty \); \hspace{1cm} (7.3)

(III) \( b^* \downarrow 0 \) for \( \sigma \downarrow 0 \). \hspace{1cm} (7.4)

Note that statement (II) is compatible with (3.31) and (7.1). The meaning of statement (III) is that, if the business has no risk, there is no need for the company to hold any surplus.

To prove these three statements, we need an alternative form for (6.2). Motivated by
\[
\frac{-s}{r} = \frac{\sqrt{\mu^2 + 2 \delta \sigma^2} + \mu}{\sqrt{\mu^2 + 2 \delta \sigma^2} - \mu}
\]

which is from applying formulas (2.14) and (2.15), we introduce a new variable

\[
z = \frac{\mu}{\sqrt{\mu^2 + 2 \delta \sigma^2}},
\]

so that

\[
\frac{-s}{r} = \frac{1+z}{1-z}.
\]

Note that \(z\) is a decreasing function of \(\sigma\), and that \(z = 0\) for \(\sigma = \infty\), and \(z = 1\) for \(\sigma = 0\).

Now, (6.2) becomes

\[
b^* = \frac{\sigma^2}{\mu} z \ln \frac{1+z}{1-z}
\]

\[
= \frac{\mu}{2 \delta} \left( \frac{1}{z} - z \right) \ln \frac{1+z}{1-z}.
\]

For \(0 \leq z < 1\), we have

\[
\ln \frac{1+z}{1-z} = \ln(1+z) - \ln(1-z)
\]

\[
= 2(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \ldots).
\]

Hence

\[
b^* = \frac{\mu}{\delta} (1 - \frac{2}{3} z^2 - \frac{2}{15} z^4 - \ldots - \frac{2}{(2n-1)(2n+1)} z^{2n} - \ldots).
\]

We see from (7.9) that \(b^*\) is a decreasing function of \(z\), \(0 \leq z < 1\), and that \(b^* \uparrow \frac{\mu}{\delta}\) for \(z \downarrow 0\), proving (7.2) and (7.3). Furthermore,

\[
\frac{2}{3} + \frac{2}{15} + \ldots + \frac{2}{(2n-1)(2n+1)} + \ldots = 1.
\]
To verify (7.10), use \( \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1} \) to write its left-hand side as a telescoping series. It now follows from (7.9) and (7.10) that \( b^* \downarrow 0 \) for \( z \uparrow 1 \), proving (7.4).

8. The Leverage Ratio

For \( 0 < x \leq b \leq b^* \), consider the ratio

\[
R(x; b) = \frac{V(x; b)}{x},
\]

(8.1)

which is the expected present value of all dividends per unit of initial capital or surplus. Note that

\[
R(x; b) = \frac{V(x; b) - V(0; b)}{x - 0}
\]

(8.2)

can be interpreted as the slope of a secant. Thus \( R(0; b) \) is defined as the derivative \( V'(0, b) \). Since \( g(x) \) is a concave function for \( x \leq b^* \), it follows from (2.11) that, for \( 0 \leq x \leq b \leq b^* \), \( V(x; b) \) is also a concave function of \( x \). Hence \( R(x; b) \) is a decreasing function of \( x \), \( 0 \leq x \leq b \leq b^* \), and

\[
R(b; b) > V'(b; b)
\]

(8.3)

by (2.11). The inequality

\[
R(x; b) > 1 \quad \text{for} \quad 0 \leq x \leq b \leq b^*
\]

(8.4)

shows that, in our model, the expected present value of all dividends generated by a barrier strategy with level \( b \), \( b \leq b^* \), is higher than the initial surplus or capital. Also, the fact that \( R(x; b) \) is a decreasing function of the initial capital \( x \) has a somewhat shocking
implication: if the investor is only interested in the leverage ratio, he would want to invest in companies with a low degree of capitalization!

For the remainder of this section, we consider $b = b^*$, the optimal barrier level, and $0 \leq x \leq b^*$. The minimum leverage ratio is

$$R(b^*; b^*) = \frac{V(b^*; b^*)}{b^*}$$

$$= 2\left[\left(\frac{1}{z} - z\right) \ln \frac{1+z}{1-z}\right]^{-1}$$

$$= \left(1 - \frac{2}{3}z^2 - \frac{2}{15}z^4 - \ldots - \frac{2}{(2n-1)(2n+1)}z^{2n} - \ldots\right)^{-1}$$

by (7.1), (7.8) and (7.9), with $z$ given by (7.5). Obviously, $R(b^*; b^*) > 1$, confirming (8.3). On the other hand, the maximum leverage ratio is

$$R(0; b^*) = V'(0; b^*)$$

$$= \frac{r-s}{re^{rb^*} - se^{sb^*}}$$

by (3.22). Substituting $b^*$ in (8.6) by (6.2) yields

$$R(0; b^*) = \frac{r-s}{r(-s/r)^{2s/(r-s)} - s(-s/r)^{2s/(r-s)}}.$$  (8.7)

Now, the ratio $-s/r$ is given by (7.6). Also, it follows from (7.6) that $r/(r-s) = (1-z)/2$ and that $s/(r-s) = -(1+z)/2$. Applying these to (8.7), we obtain

$$R(0; b^*) = \frac{2}{(1-z)\left(\frac{1+z}{1-z}\right)^{1-z} + (1+z)\left(\frac{1+z}{1-z}\right)^{-1}}$$

$$= \left(\frac{1+z}{1-z}\right)^{1-z}. $$  (8.8)
From (8.5) and (8.8), we see that both the minimal and the maximal leverage ratios, $R(b^*; b^*)$ and $R(0; b^*)$, are increasing functions of $z$, $0 \leq z < 1$. Rewriting (7.5) as

$$z = \frac{1}{\sqrt{1 + 2\delta(\sigma/\mu)^2}}, \quad (8.9)$$

we see that both $R(b^*; b^*)$ and $R(0; b^*)$ are decreasing functions of both $\delta$ and the “coefficient of variation” $\sigma/\mu$.

Formulas (8.8) and (8.9) show that $R(0; b^*) \to 1$ as $\sigma/\mu \to \infty$. Now, $R(0; b^*) \geq R(x; b^*)$ for $0 \leq x \leq b^*$. Recall (7.3) that $b^* \to \mu/\delta$ as $\sigma \to \infty$. Hence, given $\mu > 0$ and $\delta > 0$, we have $R(x; b^*) \to 1$ as $\sigma \to \infty$ for $0 \leq x \leq \mu/\delta$. In the case of infinite risk, it is not possible to have an expected total return exceeding the initial capital.

On the other hand, $R(b^*; b^*) = (\mu/\delta)b^*$ by (7.1). Recall (7.4) that $b^* \to 0$ as $\sigma \to 0$. Hence, as $\sigma \to 0$, $R(b^*; b^*) \to \infty$ and $R(0; b^*) \to \infty$. If the business has no risk, the leverage ratio becomes infinite.

**Remark** A special case of (3.25) is

$$L(b^*; b^*) = e^{-2b^*/\sigma^2} V'(0; b^*)$$

$$= e^{-2b^*/\sigma^2} \left( \frac{1+z}{1-z} \right)^{\frac{\sigma}{\mu}}, \quad (8.10)$$

by (8.6) and (8.8). Substituting $b^*$ on the right-hand side of (8.10) by (7.7) yields

$$L(b^*; b^*) = \left( \frac{1+z}{1-z} \right)^{2z} \left( \frac{1+z}{1-z} \right)^{\frac{\sigma}{\mu}}$$

$$= \left( \frac{1-z}{1+z} \right)^{\frac{\sigma}{\mu}}, \quad (8.11)$$

with $z$ given by (8.9). From this and (8.8) we see that
R(0; b*) L(b*; b*) = 1. \quad (8.12)

This identity does not seem to have an apparent interpretation.

9. The Implied Utility Function

The value of an initial capital of $x is V(x; b*), 0 \leq x \leq b*$. Thus $V(x; b*)$ can be interpreted as some sort of a utility of $x$. We shall indeed show that $V(x; b*)$ has the properties of a risk-averse utility function. In view of (2.11), we examine the function

$$g(x) = e^{rx} - e^{sx}, \quad 0 \leq x \leq b*, \quad (9.1)$$

for its properties as a utility function. Note that $g'(0) = r - s > 0$. Because $b*$ is the unique solution of equation (6.1), it follows that $g'(x) > 0$ for $0 \leq x < b*$. The associated risk aversion function is

$$\zeta(x) = \frac{-g''(x)}{g'(x)} = \frac{-r^2 e^{rx} + s^2 e^{sx}}{r e^{rx} - s e^{sx}}. \quad (9.2)$$

Observe that

$$\zeta'(x) = \frac{rs(r - s)^2 e^{rx + sx}}{(re^{rx} - se^{sx})^2} \quad (9.3)$$

after simplification. Because $rs = -2\delta/\sigma^2 < 0$, it follows that $\zeta(x)$ is a strictly decreasing function of $x$. As $\zeta(b*) = 0$ by (6.1), we have $\zeta(x) > 0$ for $0 \leq x < b*$. We note that

$$\zeta(0) = -(r + s) = \frac{2\mu}{\sigma^2} \quad (9.4)$$

by (9.2) and (3.24).
10. The Expected Utility of D

One way to take into account the randomness of D is to calculate its expected utility. Let \( u(x) \) be an appropriate risk-averse utility function. By the term “risk-averse” we mean that the function has the properties \( u'(x) > 0 \) and \( u''(x) < 0 \). Examples are the quadratic utility function with level of saturation \( s \),

\[
u(x) = x - \frac{1}{2s}x^2, \quad x < s, \tag{10.1}\]

and the exponential utility function with parameter \( \zeta > 0 \),

\[
u(x) = \frac{1}{\zeta} (1 - e^{-\zeta x}), \quad -\infty < x < \infty. \tag{10.2}\]

We are interested in the expected utility, \( E[u(D)] \).

If the barrier strategy with level \( b \) is applied, the expected utility for (10.1) and (10.2) can be calculated as follows. For the quadratic utility function (10.1), we obtain

\[
E[u(D)] = E[D] - \frac{1}{2s} E[D^2] = V_1(x; b) - \frac{1}{2s} V_2(x; b) = \frac{g_1(x)}{g_1'(b)} - \frac{1}{s} \frac{g_1(b) g_2(x)}{g_1'(b) g_2'(b)} \tag{10.3}
\]

according to (4.18). For the exponential utility function (10.2), we get

\[
E[u(D)] = \frac{1}{\zeta} (1 - E[e^{-\zeta D}]) = \frac{1}{\zeta} (1 - M(x, -\zeta; b)) = \frac{1}{\zeta} \sum_{k=1}^{\infty} (-\zeta)^{k+1} \frac{g_k(b) \ldots g_{k-2}(b) g_1(x)}{g_1'(b) \ldots g_{k-1}'(b) g_1'(b)} \tag{10.4}
\]
according to (4.19).

This leads us to the following question: what is the optimal dividend strategy, if the objective is to maximize the expected utility of the present value of the dividends until ruin? If the utility function is quadratic or exponential, the optimal strategy cannot be a barrier strategy, because if it were, we would maximize expressions (10.3) or (10.4), respectively, and the optimal value of $b$ would not depend on $x$. But this is obviously not the case. Hence maximizing the expected utility of the present value of the dividends until ruin is a new and challenging problem.

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References


**Appendix A**

For the convenience of the reader, this Appendix presents some equivalent expressions for the moment generating function of a compound geometric random variable with exponentially distributed summands. These results are used in Section 5. Let p and q
(p + q = 1) be the parameters of the geometric distribution, and let μ be the mean of the exponential distribution. Thus the moment generating function is

\[ M(y) = p \sum_{k=0}^{\infty} \left( \frac{q}{1-\mu y} \right)^k \]

\[ = \frac{p}{1 - \frac{q}{1-\mu y}}. \quad (A.1) \]

Simplification yields

\[ M(y) = \frac{p(1 - \mu y)}{p - \mu y} = \frac{1 - \mu y}{1 - \frac{\mu}{p} y}. \quad (A.2) \]

After a division we obtain

\[ M(y) = p + q \frac{1}{1 - \frac{\mu}{p} y}, \quad (A.3) \]

which shows that the underlying distribution is a mixture of the degenerate distribution at 0 and the exponential distribution with mean μ/p. We note that all steps above are reversible. Thus, if a distribution has moment generating function

\[ M(y) = \frac{1 - \alpha y}{1 - \beta y} \quad (A.4) \]

with 0 < α < β, we can conclude by comparing (A.4) with (A.2) that it is a compound geometric distribution with exponential summands whose parameters are

\[ \mu = \alpha, \quad p = \frac{\alpha}{\beta}, \quad q = \frac{\beta - \alpha}{\beta}. \quad (A.5) \]

Formula (A.1) is a special case of (12.2.9) of Bowers et al. (1997). With μ = 1, (A.3) is the same as (12.2.15) of Bowers et al. (1997). Figure 12.2.1 in Bowers et al. (1997) illustrates the distribution function. Expression (A.3) with its interpretation can
also be found in Example 4.6 of Klugman, Panjer and Willmot (1998). Furthermore, the formulas in this Appendix are useful for analyzing the distribution of the maximal loss random variable in the case where the individual claims are exponentially distributed; see Section 13.6 of Bowers et al. (1997).

Appendix B

The goal of this Appendix is to derive the discrete counterparts of identities (3.25) and (3.28) for the original De Finetti (1957) model. In this model, the surplus at time $t$ is

\[ X(t) = x + G_1 + \ldots + G_t \quad (B.1) \]

where $t = 1, 2, 3, \ldots$. Here $x = X(0)$ is the initial surplus (a positive integer), and the annual net gains $G_1, G_2, \ldots$ are independent and identically distributed random variables with

\[ \Pr(G_t = 1) = \pi, \quad \Pr(G_t = -1) = 1 - \pi, \quad (B.2) \]

where $\pi > 1/2$. Let

\[ M(t) = \max\{x, X(1), \ldots, X(t)\}. \quad (B.3) \]

The dividend barrier $b$ is a positive integer, $b \geq x$. Then the cumulative dividends by time $t$ are given by

\[ D(t) = \begin{cases} 0 & \text{if } M(t) \leq b, \\ M(t) - b & \text{if } M(t) > b. \end{cases} \quad (B.4) \]

Let $d_t$ denote the dividend paid at time $t$. Thus

\[ D(t) = d_1 + d_2 + \ldots + d_t. \quad (B.5) \]

Note that the random variable $d_t$ assumes only the values 0 or 1. Let $0 < v < 1$ be a discount factor,

\[ T = \min\{t \geq 0 \mid X(t) - D(t) = 0\} \quad (B.6) \]

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be the time of ruin, and

\[ V(x; b) = E\left[ \sum_{t=1}^{T} v^t d_t \mid X(0) = x \right] \quad (B.7) \]

be the expectation of the present value of the dividends until ruin.

Let \( x = 1, 2, \ldots, b - 1 \). By distinguishing whether \( G_1 = 1 \) or \( G_1 = -1 \), we see that

\[ V(x; b) = v\{ \pi V(x + 1; b) + (1 - \pi)V(x - 1; b) \}. \quad (B.8) \]

Thus, as a function of \( x \), \( V(x; b) \) is a linear combination of \( r^x \) and \( s^x \), where \( 0 < s < 1 < r \) are the solutions of the indicial equation

\[ v\pi \zeta^2 - \zeta + v(1 - \pi) = 0. \quad (B.9) \]

The coefficients of this linear combination are determined from the boundary conditions:

\[ V(0; b) = 0, \quad (B.10) \]
\[ V(b; b) = v\{ \pi[1 + V(b; b)] + (1 - \pi)V(b - 1; b) \}. \quad (B.11) \]

It follows that

\[ V(x; b) = \frac{r^x - s^x}{(r - 1)r^b - (s - 1)s^b}, \quad (B.12) \]

\( x = 0, 1, \ldots, b \). In particular,

\[ V(1; b) = \frac{r - s}{(r - 1)r^b - (s - 1)s^b}. \quad (B.13) \]

Let

\[ L(x; b) = E[v^T \mid X(0) = x] \quad (B.14) \]

denote the expected discounted value of a payment of 1 at the time of ruin. As a function of \( x \), \( L(x; b) \) satisfies a difference equation like (B.8). Hence \( L(x; b) \) is also a linear combination of \( r^x \) and \( s^x \). The boundary conditions are now:

\[ L(0; b) = 1, \quad (B.15) \]
\[
L(b; b) = v\{\pi L(b; b) + (1 - \pi)L(b - 1; b)\}. \tag{B.16}
\]

It follows that
\[
L(x; b) = \frac{(r - 1)s^{-b} - (s - 1)r^{-b}}{(r - 1)s^{-b} - (s - 1)r^{-b}}, \quad (B.17)
\]
\[x = 0, 1, \ldots, b. \text{ In particular,} \]
\[
L(b; b) = (r - s) V_{1; b}, \quad (B.18)
\]
by (B.13). From (B.9) we gather that \(rs = (1 - \pi) / \pi\). Hence
\[
L(b; b) = \left(\frac{1 - \pi}{\pi}\right)^b V(1; b), \quad (B.19)
\]
which is the counterpart of (3.25). Now, it follows from (B.14) and (B.7) that
\[
L(x; b) = \sum_{t=1}^{\infty} v^t \Pr(T = t \mid X(0) = x), \quad (B.20)
\]
\[
V(x; b) = \sum_{t=1}^{\infty} v^t \Pr(d_t = 1 \text{ and } T > t \mid X(0) = x). \quad (B.21)
\]
Because (B.19) holds for all \(v \in (0, 1)\), we conclude that, for \(t = 1, 2, 3, \ldots\),
\[
\Pr(T = t \mid X(0) = b) = \left(\frac{1 - \pi}{\pi}\right)^b \Pr(d_t = 1 \text{ and } T > t \mid X(0) = 1). \quad (B.22)
\]
The probabilistic identities (B.22) correspond to (3.28).